JOURNAL OF APPROXIMATION THEORY 33, 224–235 (1981)

Local Nikolskii Constants for Positive Linear Operators

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Communicated by G. Meinardus

Received September 25, 1980

1. INTRODUCTION

Let X, $\hat{X}$ be two subsets of $\mathbb{R}$ with $\hat{X} \subset X$ and let $(L_n)_{n\in\mathbb{N}}$ be a sequence of linear positive operators defined on the domain

$$C_M(X) := \{f \in C(X)\text{: there exist constants } A(f),\ B(f) \in \mathbb{R}^+ \text{ and } m(f) \in \mathbb{N} \text{ such that } |f(t)| \leqslant A(f) + B(f)\,|t|^{m(f)} \text{ for all } t \in X\}$$

into $C(\hat{X})$. Further we assume that the sequence $(L_n)_{n\in\mathbb{N}}$ has the property

$$\lim_{n\to\infty} L_n(f; x) = f(x), \qquad f \in C_M(X),\ x \in \hat{X}.$$

Let now $S \subset C_M(X)$ be a class of functions, then

$$\Delta(L_n; S; x) := \sup_{f\in S} |L_n(f; x) - f(x)|, \qquad x \in \hat{X},$$

is called the local measure of approximation of the class S by the operator L_n.

If there exists a numerical sequence $\Psi_n(L; S) \to 0$ as $n \to \infty$ such that

$$\Delta(L_n; S; x) = C(L; S; x)\, \Psi_n(L; S) + o_x(\Psi_n(L; S)), \qquad n \to \infty,\ x \in \hat{X},$$

then $C(L; S; x)$ is designated the local best asymptotic constant corresponding to the order $\Psi_n(L; S)$ of approximation of the class S by the operator L_n. When the class S coincides with a Lipschitz class, the constant $C(L; S; x)$ will be called a local Nikolskii constant.

* This paper is part of the author's dissertation.

0021-9045/81/110224-12$02.00/0

In this paper S will always be one of the following three classes

$$\begin{aligned}\operatorname{Lip}(C(X);\alpha;1) &:= \{f \in C(X): |f(x+h)-f(x)| \leqslant |h|^{\alpha}, \\ &\qquad \forall x, x+h \in X\}, \qquad 0<\alpha \leqslant 1; \\ \operatorname{Lip}^*(C(X);\alpha;2) &:= \{f \in C(X): |f(x+h)-2f(x)+f(x)+f(x-h)| \leqslant 2\,|h|^{\alpha}, \\ &\qquad \forall x, x \pm h \in X\}, \qquad 0<\alpha \leqslant 2; \\ W^{(1)}(C(X);\alpha;1) &:= \{f \in C^1(X): f' \in \operatorname{Lip}(C(X);\alpha;1), \qquad 0<\alpha \leqslant 1.\end{aligned}$$

Another aspect of this paper is to study the approximation of higher order by the operator L_n. For that purpose we consider the quantity

$$\Delta(L_n;\alpha;q;x) := \sup_{f \in W^{(q)}(C(X);\alpha;1)} \left| L_n\left(f(t) - \sum_{k=0}^{q} \frac{f^{(k)}(x)}{k!}(t-x)^k; x\right)\right|$$

with

$$W^{(q)}(C(X);\alpha;1) := \{f \in C^q(X): f^{(q)} \in \operatorname{Lip}(C(X);\alpha;1)\}, \\ 0<\alpha \leqslant 1, \quad q \in \mathbb{N}.$$

2. The Local Measure of Approximation

In this section we shall see that it is sometimes useful to change from the given sequence $(L_n)_{n\in\mathbb{N}}$ to a new sequence $(\tilde{L}_n)_{n\in\mathbb{N}}$ of positive linear operators, which are for suitable functions g formally defined by

$$\tilde{L}_n(g;x) := \tfrac{1}{2}\{L_n(g;x) + L_n(g(2x-t);x)\}, \qquad x \in \hat{X}. \tag{2.1}$$

By means of (2.1) we obtain from the assumptions about the sequence $(L_n)_{n\in\mathbb{N}}$.

Proposition 2.1. *Let $X' := X \cup \{2x-t: x \in \hat{X}, t \in X\}$, then $(\tilde{L}_n)_{n\in\mathbb{N}}$ is a sequence of positive linear operators from $C_M(X')$ in $C(\hat{X})$ with the property*

$$\lim_{n\to\infty} \tilde{L}_n(g;x) = g(x), \qquad g \in C_M(X'), \quad x \in \hat{X}.$$

In the proofs of our main results we shall require the following Lemma of Rathore.

Lemma 2.2. *Let C, D be two subsets of $\mathbb{R}$ with $C \subset D$. Then*

(a) *For every fixed $x \in C$ and $0 < a \leqslant 1$ the function $f_{\alpha,x}(t) := |t-x|^{\alpha}$, $t \in D$ is of class* $\operatorname{Lip}(C(B);\alpha;1)$.

(b) *For every fixed* $x \in C$ *and* $0 < \alpha \leqslant 1$ *the function* $g_{\alpha,x}(t) := 2^{\alpha-1}\operatorname{sgn}(t-x)\,|t-x|^{\alpha}$, $t \in D$ *is of class* $\mathrm{Lip}(C(B); \alpha; 1)$, *where the Lipschitz constant* 1 *is best possible.*

A complete proof of Lemma 2.2 is given in [2; p. 18 ff.] (compare also Rathore [5; p. 51]).

THEOREM 2.3. (a) *If*

$$L_n(1; x) = 1, \qquad x \in \hat{X},$$

then we have for all $x \in \hat{X}$ *and* $0 < \alpha \leqslant 1$

$$\Delta(L_n; \mathrm{Lip}(C(X); \alpha; 1); x) = L_n(|t-x|^{\alpha}; x). \tag{2.2}$$

(b) *If*

$$L_n(1; x) = 1, \qquad x \in \hat{X} \qquad \textit{and} \qquad L_n(t; x) = x,\ x \in \hat{X},$$

then we have for all $x \in \hat{X}$

$$\frac{2^{\alpha-1}}{1+\alpha} L_n(|t-x|^{\alpha+1}; x) \leqslant \Delta(L_n; W^{(1)}(C(X); \alpha; 1); x) \leqslant \frac{1}{1+\alpha} L_n(|t-x|^{\alpha+1}; x), \qquad 0 < \alpha \leqslant 1, \tag{2.3}$$

$$\Delta(L_n; W^{(1)}(C(X); 1; 1); x) = \tfrac{1}{2} L_n((t-x)^2; x). \tag{2.4}$$

(c) *If*

$$L_n(1; x) = 1, \qquad x \in \hat{X},$$

then we have for all $x \in X$

$$\Delta(\tilde{L}_n; \mathrm{Lip}(C(X); \alpha; 1); x) = L_n(|t-x|^{\alpha}; x), \qquad 0 < \alpha \leqslant 1, \tag{2.5}$$

$$\Delta(\tilde{L}_n; \mathrm{Lip}^*(C(X); \alpha; 2); x) = L_n(|t-x|^{\alpha}; x), \qquad 0 < \alpha \leqslant 2, \tag{2.6}$$

$$\Delta(\tilde{L}_n; W^{(1)}(C(X); \alpha; 1); x) = \frac{2^{\alpha-1}}{1+\alpha} L_n(|t-x|^{\alpha+1}; x), \qquad 0 < \alpha \leqslant 1. \tag{2.7}$$

THEOREM 2.4. *For* $x \in \hat{X}$ *we have*

$$\Delta(L_n; \alpha; q; x) = \Delta(\tilde{L}_n; \alpha; q; x) = \frac{1}{(1+\alpha)_q} L_n(|t-x|^{\alpha+q}; x), \qquad 0 < \alpha \leqslant 1,\ q \in \mathbb{N}\ \textit{even}; \tag{2.8}$$

$$\frac{2^{\alpha-1}}{(1+\alpha)_q} L_n(|t-x|^{\alpha+q}; x) \leqslant \Delta(L_n; \alpha; q; x)$$

$$\leqslant \frac{1}{(1+\alpha)_q} L_n(|t-x|^{\alpha+q}; x), \qquad 0<\alpha \leqslant 1,\ q \in \mathbb{N} \textit{ odd}; \quad (2.9)$$

$$\Delta(L_n; 1; q; x) = \frac{1}{(q+1)!} L_n(|t-x|^{q+1}; x), \qquad q \in \mathbb{N} \textit{ odd}; \quad (2.10)$$

$$\Delta(\tilde{L}_n; \alpha; q; x) = \frac{2^{\alpha-1}}{(1+\alpha)_q} L_n(|t-x|^{\alpha+q}; x), \qquad 0<\alpha \leqslant 1,\ q \in \mathbb{N} \textit{ odd}, \quad (2.11)$$

where $(1+\alpha)_q := \prod_{k=1}^{q} (k+\alpha)$.

Proof of Theorem 2.3. (a) For $f \in \mathrm{Lip}(C(X); \alpha; 1)$ and $x \in \hat{X}$ we obtain

$$|L_n(f; x) - f(x)| \leqslant L_n(|f(t)-f(x)|; x) \leqslant L_n(|t-x|^{\alpha}; x).$$

Now we consider for every fixed $x \in \hat{X}$ the function $f_{\alpha,x}(t) := |t-x|^{\alpha}$, $t \in X$ and by means of Lemma 2.2 we get

$$\Delta(L_n; \mathrm{Lip}(C(X); \alpha; 1); x) \geqslant |L_n(f_{\alpha,x}; x) - f_{\alpha,x}(x)|$$
$$= L_n(|t-x|^{\alpha}; x).$$

(b) For $f \in W^{(1)}(C(X); \alpha; 1)$ and $x \in \hat{X}$ we have

$$|f(t)-f(x)-f'(x)(t-x)| \leqslant \left| \int_x^t |f'(z)-f'(x)|\, dz \right|$$
$$\leqslant \frac{1}{1+\alpha} |t-x|^{\alpha}$$

and thus

$$|L_n(f; x) - f(x)| \leqslant |L_n(f(t)-f(x)-f'(x)(t-x); x)|$$
$$\leqslant \frac{1}{1+\alpha} L_n(|t-x|^{\alpha+1}; x).$$

The first part of inequality (2.2) follows from Lemma 2.2, while considering for fixed $x \in \hat{X}$ the function

$$g_{\alpha,x}(t) := \frac{2^{\alpha-1}}{1+\alpha} |t-x|^{\alpha+1}, \qquad t \in X.$$

(c) From (2.1) we get

$$|L_n(f;x)-f(x)| = \tfrac{1}{2}|L_n(f(t)-2f(x)+f(2x-t);x)|$$
$$\leqslant \tfrac{1}{2}L_n(|f(t)-2f(x)+f(2x-t)|;x).$$

Moreover for $f \in \text{Lip}(C(X');\alpha;1)$, $0<\alpha\leqslant 1$, or $f\in \text{Lip}^*(C(X');\alpha;2)$, $0<\alpha\leqslant 2$, we have

$$|f(t)-2f(x)+f(2x-t)| \leqslant 2\,|t-x|^\alpha,$$

whereas for $f \in W^{(1)}(C(X');\alpha;1)$ there holds

$$|f(t)-2f(x)+f(2x-t)| = \left|\int_0^{t-x}(f(x+u)-f(x-u))\,du\right|$$
$$\leqslant \left|\int_0^{t-x}|2u|^\alpha\,du\right| = \frac{2^\alpha}{1+\alpha}|t-x|^{\alpha+1}.$$

On the other hand the function $f_{\alpha,x}(t):=|t-x|^\alpha$, $t\in X'$, is an extreme function for the class $\text{Lip}(C(X');\alpha;1)$, $0<\alpha\leqslant 1$, and the class $\text{Lip}^*(C(X');\alpha;2)$, $0<\alpha\leqslant 2$.

Moreover the function

$$g_{\alpha,x}(t):=\frac{2^{\alpha-1}}{1+\alpha}|t-x|^{\alpha+1}, \qquad t\in X',$$

is an extreme function for the class $W^{(1)}(C(X');\alpha;1)$, $0<\alpha\leqslant 1$ (cf. Lemma 2.2). ∎

Proof of Theorem 2.4. (a) If $f\in W^{(q)}(C(X);\alpha;1)$ and $x\in \hat{X}$, then

$$f(t)-\sum_{k=0}^{q}\frac{f^{(k)}(x)}{k!}(t-x)^k = \int_x^t\int_x^{t_q}\cdots\int_x^{t_2}\{f^{(q)}(t_1)-f^{(q)}(x)\}\,dt_1\cdots dt_q \tag{2.12}$$

and hence

$$\left|f(t)-\sum_{k=0}^{q}\frac{f^{(k)}(x)}{k!}(t-x)^k\right|$$
$$\leqslant \left|\int_x^t\int_x^{t_q}\cdots\int_x^{t_2}|f^{(q)}(t_1)-f^{(q)}(x)|\,dt_1\cdots dt_q\right| \tag{2.13}$$
$$\leqslant \left|\int_x^t\int_x^{t_q}\cdots\int_x^{t_2}|t_1-x|^\alpha\,dt_1\cdots dt_q\right| = \frac{1}{(1+\alpha)_q}|t-x|^{\alpha+q}.$$

From (2.13) follows by the positivity of L_n

$$\Delta(L_n; \alpha; q; x) \leqslant \frac{1}{(1+\alpha)_q} L_n(|t-x|^{\alpha+q}; x).$$

Now for even $q \in \mathbb{N}$ the function

$$h_{\alpha,q,x}(t) := \frac{1}{(1+\alpha)_q} |t-x|^{\alpha+q}, \qquad t \in X,$$

is an extreme function of the class $W^{(q)}(C(X); \alpha; 1)$. For odd $q \in \mathbb{N}$ the first inequality of (2.9) follows from Lemma 2.2, while considering the function

$$k_{\alpha,q,x}(t) := \frac{2^{\alpha-1}}{(1+\alpha)_q} |t-x|^{\alpha+q}, \qquad t \in X.$$

(b) Let $f \in W^{(q)}(C(X'); \alpha; 1)$ and $x \in \hat{X}$. Then, putting $t_i := 2x - z_i$ $(i = 1, \ldots, q)$, $t = 2x - u$, we obtain from (2.12)

$$\begin{aligned} f(2x-u) &- \sum_{k=0}^{q} \frac{f^{(k)}(x)}{k!} (x-u)^k \\ &= \int_x^u \int_x^{z_q} \cdots \int_x^{z_2} (-1)^q \{f^{(q)}(2x-z_1) - f^{(q)}(x)\} \, dz_1 \cdots dz_q. \end{aligned} \tag{2.14}$$

With respect to (2.1) we get from (2.12) and (2.14) for even q

$$\begin{aligned} &\left| \tilde{L}_n \left(f(t) - \sum_{k=0}^{q} \frac{f^{(k)}(x)}{k!} (t-x)^k; x \right) \right| \\ &\quad = \frac{1}{2} \left| L_n \left(f(t) - \sum_{k=0}^{q} \frac{f^{(k)}(x)}{k!} (t-x)^k; x \right) \right. \\ &\qquad \left. + L_n \left(f(2x-t) - \sum_{k=0}^{q} \frac{f^{(k)}(x)}{k!} (x-t)^k; x \right) \right| \\ &\quad \leqslant \frac{1}{2} L_n \left(\left| \int_x^t \int_x^{t_q} \cdots \int_x^{t_2} |f^{(q)}(t_1) - 2f^{(q)}(x) + f^{(q)}(2x-t_1)| \, dt_1 \cdots dt_q \right|; x \right) \\ &\quad \leqslant \frac{1}{2} L_n \left(\left| \int_x^t \int_x^{t_q} \cdots \int_x^{t_2} 2|t_1 - x|^\alpha \, dt_1 \cdots dt_q \right|; x \right) \\ &\quad = \frac{1}{(1+\alpha)_q} L_n(|t-x|^{\alpha+q}; x). \end{aligned}$$

Analogously we get for odd q

$$\left| \tilde{L}_n \left(f(t) - \sum_{k=0}^{q} \frac{f^{(k)}(x)}{k!} (t-x)^k; x \right) \right| \leqslant \frac{2^{\alpha-1}}{(1+\alpha)_q} L_n(|t-x|^{\alpha+q}; x).$$

Now, for even q, $h_{\alpha,q,x}$ is again an extreme function referring to the quantity $\Delta(\tilde{L}_n; \alpha; q; x)$. But this time also $k_{\alpha,q,x}$ is an extreme function referring to $\Delta(\tilde{L}_n; \alpha; q; x)$ for odd q. ∎

Remark. The results of Theorems 2.3 and 2.4 generalize and improve some special results of Rathore [5; pp. 51, 54, 66; 6; 7]. (Compare also Lehnhoff [1; pp. 129, 131].)

3. An Asymptotic Estimation for $L_n(|t-x|^\beta; x)$

As we have seen in Section 2, in the theory of Nikolskii constants an important role is played by an asymptotic estimation for $L_n(|t-x|^\beta; x)$.

(I) For positive linear operators of form

$$L_n(f; x) := \int_X f(u)\, h_n(x, u)\, du, \qquad x \in \hat{X},\ n \in \mathbb{N}$$

with functions $h_n : \hat{X} \times X \to \mathbb{R}^+$ the following Theorem holds.

THEOREM 3.1. *Suppose, there exist a sequence $(\chi_n)_{n\in\mathbb{N}}$ of positive numbers increasing to infinity with n, a positive function Φ from $\hat{X}$ into $\mathbb{R}$, a positive integer m and a real α, $\alpha < 1/2$ such that the following conditions are satisfied.*

(i) *For $x \in (\hat{X})^0$ the asymptotic relation*

$$h_n(x, t) \cong \sqrt{\frac{\chi_n}{2\pi\Phi(x)}} \exp\left[\frac{-\chi_n}{2\Phi(x)}(t-x)^2\right], \qquad n \to \infty, \tag{3.1}$$

holds uniformly for all $t \in A_n(x) := \{t \in \mathbb{R} : |t-x| < (\chi_n)^{-\alpha}\}$.

(ii) *For $x \in (\hat{X})^0$ we have*

$$L_n((t-x)^{2m}; x) = O_x((\chi_n)^{-m}), \qquad n \to \infty.$$

Then for all $x \in (\hat{X})^0$ and all $0 < \beta < 2m$ we get

$$L_n(|t-x|^\beta; x) \cong \frac{\Gamma((\beta+1)/2)}{\sqrt{\pi}} \left(\frac{2\Phi(x)}{\chi_n}\right)^{\beta/2}, \qquad n \to \infty.$$

(*In this connection $(\hat{X})^0$ means the interior of $\hat{X}$!*)

Proof. It is clear that $\lim_{n\to\infty}(\chi_n)^{-\alpha} = 0$.

Thus for every $x \in (\hat{X})^0$ there exists a positive integer $n_0(x)$ with the property that $A_n(x) \subset X$ for all $n \geqslant n_0(x)$.

Hence we have

$$L_n(|t-x|^\beta; x) = I_1(x) + I_2(x)$$

with

$$\begin{aligned} I_1(x) &:= \int_{A_n(x)} |u-x|^\beta \, h_n(x,u)\, du \\ &\cong \sqrt{\frac{\chi_n}{2\pi\Phi(x)}} \int_{A_n(x)} |u-x|^\beta \exp\left[\frac{\chi_n}{2\Phi(x)}(u-x)^2\right] du \\ &= \frac{1}{\sqrt{\pi}}\left(\frac{2\Phi(x)}{\chi_n}\right)^{\beta/2} \int_{|v| < (\chi_n)^{1/2-\alpha}/\sqrt{2\Phi(x)}} |v|^\beta \exp(-v^2)\, dv \\ &\cong \frac{1}{\sqrt{\pi}}\left(\frac{2\Phi(x)}{\chi_n}\right)^{\beta/2} \int_{-\infty}^{\infty} |v|^\beta \exp(-v^2)\, dv \\ &= \frac{\Gamma((\beta+1)/2)}{\sqrt{\pi}}\left(\frac{2\Phi(x)}{\chi_n}\right)^{\beta/2} \end{aligned}$$

and

$$\begin{aligned} I_2(x) &:= \int_{\substack{|u-x| \geqslant (\chi_n)^{-\alpha} \\ u \in X}} |u-x|^\beta h_n(x,u)\, du \\ &\leqslant (\chi_n)^{2\alpha(m-\beta/2)} \int_{\substack{|u-x| \geqslant (\chi_n)^{-\alpha} \\ u \in X}} (u-x)^{2m} h_n(x,u)\, du \\ &\leqslant (\chi_n)^{2\alpha(m-\beta/2)} L_n((t-x)^{2m}; x) \\ &= O_x((\chi_n)^{-m+2\alpha(m-\beta/2)}) = o_x((\chi_n)^{-\beta/2}) \qquad \text{for} \quad \beta < 2m. \quad \blacksquare \end{aligned}$$

(II) For positive linear operators of form

$$L_n(f; x) := \sum_{k \in I} \Psi_{k,n}(x)\, f(a_{k,n}), \qquad x \in \hat{X},$$

with an index set $I \subset \mathbb{Z}$, a sequence $(\Psi_{k,n})_{k\in\mathbb{Z}, n\in\mathbb{N}}$ of positive functions from $\hat{X}$ into $\mathbb{R}$ and a sequence $(a_{k,n})_{k\in\mathbb{Z}, n\in\mathbb{N}}$ of points, which satisfy the conditions

$$a_{k,n} \in X \qquad \text{for} \quad k \in I,\ n \in \mathbb{N}$$

$$a_{\nu,n} < a_{\mu,n} \qquad \text{for} \quad \nu, \mu \in \mathbb{Z},\ \nu < \mu,$$

we obtain the following theorem.

THEOREM 3.2. *Suppose, there exist a sequence $(\chi_n)_{n\in\mathbb{N}}$ of positive numbers increasing to infinity with n, a positive function Φ from $\hat{X}$ into $\mathbb{R}$, a positive integer m and a real α, $\alpha < 1/2$, such that the following conditions are valid.*

(i) *For every $x \in (\hat{X})^0$ exists a positive integer $n_0(x)$ with the property that*

$$A_n(x) := \{k \in \mathbb{Z} : |a_{k,n} - x| < (\chi_n)^{-\alpha}\} \subset I \qquad \text{for all } n \geqslant n_0(x).$$

(ii) *Uniformly for all $k \in A_n(x)$ there holds*

$$|a_{k\pm 1,n} - a_{k,n}| = O((\chi_n)^{-1}), \qquad n \to \infty.$$

(iii) *For $x \in (\hat{X})^0$ the asymptotic relation*

$$\Psi_{k,n}(x) \cong (a_{k+1,n} - a_{k,n}) \sqrt{\frac{\chi_n}{2\pi\Phi(x)}} \exp\left[\frac{-\chi_n}{2\Phi(x)}(a_{k,n} - x)^2\right], \tag{3.2}$$

$n \to \infty$, holds uniformly for all $k \in A_n(x)$.

(iv) *For $x \in (\hat{X})^0$ we have*

$$L_n((t-x)^{2m}; x) = O_x((\chi_n)^{-m}), \qquad n \to \infty.$$

Then for all $x \in (\hat{X})^0$ and all $0 < \beta < 2m$ we get

$$L_n(|t-x|^\beta; x) \cong \frac{\Gamma((\beta+1)/2)}{\sqrt{\pi}} \left(\frac{2\Phi(x)}{\chi_n}\right)^{\beta/2}, \qquad n \to \infty.$$

Proof. Applying (iii) we obtain

$$\begin{aligned} I_1(x) &:= \sum_{k \in A_n(x)} |a_{k,n} - x|^\beta \, \Psi_{k,n}(x) \\ &\cong \sqrt{\frac{\chi_n}{2\pi\Phi(x)}} \sum_{k\in A_n(x)} |a_{k,n} - x|^\beta \, (a_{k+1,n} - a_{k,n}) \\ &\quad \times \exp\left[\frac{-\chi_n}{2\Phi(x)}(a_{k,n} - x)^2\right] \qquad (3.3) \\ &\cong \sqrt{\frac{\chi_n}{2\pi\Phi(x)}} \sum_{k\in A_n(x)} |a_{k,n} - x|^\beta \int_{a_{k,n}}^{a_{k+1,n}} \\ &\quad \times \exp\left[\frac{-\chi_n}{2\Phi(x)}(u - x)^2\right] du, \qquad n \to \infty. \qquad (3.4) \end{aligned}$$

To deduce (3.4) from (3.3), it is sufficient to remark that the integrand in (3.3) differs from

$$\exp\left[\frac{-\chi_n}{2\Phi(x)}(a_{k,n}-x)^2\right]$$

by the factor

$$\exp\left[\frac{-\chi_n}{2\Phi(x)}(a_{k,n}-u)(a_{k,n}+u-2x)\right]$$

which converges to 1 uniformly in k and u, if $a_{k,n} \leqslant u \leqslant a_{k+1,n}$ and $k \in A_n(x)$.

(a) $0 < \beta \leqslant 1$

From the inequality

$$|u-x|^\beta - |a_{k,n}-u|^\beta \leqslant |a_{k,n}-x| \leqslant |u-x|^\beta + |a_{k,n}-u|^\beta$$

we obtain by means of condition (ii)

$$|a_{k,n}-x|^\beta = |u-x|^\beta + O((\chi_n)^{-\beta}), \qquad n \to \infty, \tag{3.5}$$

Uniformly in $k \in A_n(x)$ and $a_{k,n} \leqslant u \leqslant a_{k+1,n}$.

Hence

$$I_1(x) \cong \sqrt{\frac{\chi_n}{2\pi\Phi(x)}}\left(\int_{a_{k',n}}^{a_{k''',n}} |u-x|^\beta \exp\left[\frac{-\chi_n}{2\Phi(x)}(u-x)^2\right] du \right.$$
$$\left. + O((\chi_n)^{-\beta}) \int_{a_{k',n}}^{a_{k''',n}} \exp\left[\frac{-\chi_n}{2\Phi(x)}(u-x)^2\right] du\right), \qquad n \to \infty,$$

where $k''' = k'' + 1$ and k'' denotes the greatest as well as k' the smallest k, for which the condition $k \in A_n(x)$ is satisfied. Putting

$$v := \sqrt{\frac{\chi_n}{2\Phi(x)}}(x-u),$$

we have

$$I_1(x) \cong \frac{1}{\sqrt{\pi}}\left(\left(\frac{2\Phi(x)}{\chi_n}\right)^{\beta/2} \int_{a_n(x)}^{b_n(x)} |v|^\beta \exp(-v^2)\, dv \right.$$
$$\left. + O((\chi_n)^{-\beta}) \int_{a_n(x)}^{b_n(x)} \exp(-v^2)\, dv\right)$$

with

$$a_n(x) = \sqrt{\frac{\chi_n}{2\Phi(x)}}\,(x - a_{k'',n}) \to -\infty,$$

$$b_n(x) = \sqrt{\frac{\chi_n}{2\Phi(x)}}\,(x - a_{k',n}) \to +\infty, \qquad n \to \infty.$$

Thus

$$I_1(x) \cong \frac{\Gamma((\beta+1)/2)}{\sqrt{\pi}} \left(\frac{2\Phi(x)}{\chi_n}\right)^{\beta/2}, \qquad n \to \infty. \tag{3.6}$$

(b) $\beta > 1$.

For $\beta > 1$ there exist $q \in \mathbb{N}$ and $\gamma \in \mathbb{R}^+$, $0 < \gamma < 1$ with $\beta = \gamma + q$. Also

$$\begin{aligned} |a_{k,n} - x|^q &= \left| \sum_{i=0}^{q} \binom{q}{i} (u-x)^i (a_{k,n} - u)^{q-i} \right| \\ &\lesseqgtr |u - x|^q \pm \sum_{i=0}^{q-1} \binom{q}{i} |u-x|^i\, |a_{k,n} - u|^{q-i}. \end{aligned} \tag{3.7}$$

By means of (3.5) and (3.7) we can verify the asymptotic relation (3.6) also for $\beta > 1$.

Moreover we obtain from condition (iv) for all $\beta < 2m$.

$$I_2(x) = \sum_{\substack{k \in I \\ |a_{k,n} - x| \geq (\chi_n)^{-\alpha}}} |a_{k,n} - x|^\beta \Psi_{k,n}(x) = o_x((\chi_n)^{-\beta/2}). \tag{3.8}$$

From (3.6) and (3.8) the result of Theorem 3.2 follows by means of condition (i). ∎

Remarks. (1) Theorem 3.1 and Theorem 3.2 were formulated and proved analogously to a special result of Rathore [5; pp. 40–46, 52].

(2) With the general theory of this paper the author has obtained local Nikolskii constants for many of the operator sequences (e.g., Baskakov operators, generalized Meyer–König and Zeller operators, Gamma operators and Beta operators etc. cf. [2, 3]).

References

1. H.-G. Lehnhoff, "Über Approximationsmasse singulärer Integrale," Diplomarbeit, Universität Dortmund, 1977.
2. H.-G. Lehnhoff, "Lokale Approximationsmaße und Nikolskii-Konstanten für positive lineare Operatoren," Dissertation, Universität Dortmund, 1979.

3. H.-G. LEHNHOFF, Local Nikolskii constants for a special class of Baskakov operators, to appear.
4. G. G. LORENTZ, "Bernstein Polynomials," Univ. of Toronto Press, Toronto, 1953.
5. R. K. S. RATHORE, "Linear Combinations of Linear Positive Operators and Generating Relations in Special Functions," Dissertation, Indian Institute of Technology, Delhi, 1973.
6. R. K. S. RATHORE, Lipschitz–Nikolskii constants for the Gamma operators of Müller, *Math. Z.* **141** (1975), 193–198.
7. R. K. S. RATHORE, Lipschitz–Nikolskii constants and asymptotic simultaneous approximation of the M_n-operators, *Aequationes Math.* **18** (1978), 206–217.
8. P. C. SIKKEMA, Über eine Verallgemeinerung zweier Sätze von G. G. Lorentz, *Indag. Math.* **38** (1976), 76–90.